Numerical Properties of Stochastic Linear Quadratic Model with Applications in Finance

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Abstract: The aim of this paper is to consider the characteristics of the numerical equilibrium solution of the stochastic linear quadratic models (SLQ) along with possible applications in financial modelling. The purpose of this approach is to find feedback control function that maximizes the portfolio value keeping the condition that stock prices are modeled by stochastic differential equation.

Two iterations – the Newton iteration and the Lyapunov iteration for solving the generalized algebraic Riccati equation, associated with the stochastic linear-quadratic problem in an infinite time horizon are discussed. We compare these iterations with the approach based on the solution to a semidefinite programming problem. Finally, in order to demonstrate the efficiency of the proposed algorithms, computational examples are provided and numerical effectiveness of the considered algorithms is commented.

Key Words: Stochastic linear-quadratic control; Generalized algebraic Riccati equation; Positive definite solution; Linear matrix inequality; Portfolio optimization

Stochastic Linear Quadratic Model

Let us consider the following SLQ model (Yao, Zhang & Zhou, 2006):

$$\min \quad E\int_{t_0}^{\infty}(y(t)^T Qy(t)+u(t)^T Ru(t))dt$$

(1) $$dy(t)=[Ay(t)+Bu(t)]dt+\sum_{j=1}^{n}[C_j y(t)+D_j u(t)]dW_j(t)\,,$$

$$y(0)=y_0\,.$$

In the above control problem Q, R, A, B, C_j and D_j for j=1, ..., n are constant matrices with appropriate dimensions, $y(.)$ denotes the state variable, and $u(.)$ the control. The model is defined on a filtered probability space (Ω, F, F_t, P) involving an n-dimensional standard Brownian motion W(t).

The solution of the SLQ problem is related to a stochastic algebraic Riccati equation which is a result of the indefiniteness of the linear quadratic model.

Recently a computational approach to stochastic algebraic Riccati equation is developed based on a semidefinite programming problem over linear matrix inequalities (LMI). Many authors have considered a semidefinite programming problem as an unifying approach to stochastic linear quadratic problem in the absence of the positive definiteness (semidefiniteness) of the cost matrices R and Q.

The introduced model (1) can be directly related to portfolio oprimization problem (Yao, Zhang & Zhou, 2006), where the control of a portfolio affects not only the average return of the portfolio but also its volatility.

Consider m listed stocks that are constituent of a market index. Assume that the price of each stock $S_i(t)$, $i=1,2,...,m$ follows the multi-dimensional GBM:

$$dS_i(t)=b_i S_i(t)dt+\sum_{j=1}^{m}\sigma_{ij}S_i(t)dW_j(t), \qquad S_i(0)=S_{i0},$$

where $W(t)=(W_1(t)\,,\,.\,.\,.,\;W_m(t))^T$ is an m-dimensional standard Brownian motion (with $t\in[0,\infty)$ and $W(0)=0$), defined on a filtered probability space (Ω, F, F_t, P).

Further assume that there is a risk less asset, the price of which is $S_0(t)$:

$$dS_0(t)=rS_0(t)dt, \quad S_0(0)=S_{00}\,.$$

Given a portfolio of n (n $\leq$ m) stocks out of the m constituent stocks, our objective is to control the investment of a given wealth initially values at x_0, among the n stocks and the bond, via dynamic asset allocation, in such a way that the performance of the investment follows as closely as possible a pre-specified, deterministic, continuously compounded growth trajectory, $x_0 e^{\mu t}$ (where $\mu > 0$ is a given parameter representing the growth factor) over a long time horizon. Here, the number of stocks in the portfolio n is a typically much smaller that m, the number of stocks in the market index. Thus we are essentially dealing with a portfolio selection problem in an incomplete market. Assume that the first n of the m stocks have been selected for the portfolio.

Let $\pi_i(t)$, $i = 1,2,...,n$ denote the wealth invested in stock i at time t. That is $\pi(.) = (\pi_1(t), \ldots, \pi_n(t))^T$ is the composition of the stock portfolio at time t, and it is called a (continuous - time) portfolio. In control parlance, $\pi(.)$ is the control. We say the portfolio or control is admissible if $\pi(.)$ belongs to $L_F^2(R^n)$, the space of all R^n-valued, F_t-adapted measurable processes satisfying $E\int_0^{\infty} \| \pi(t) \|^2 dt < +\infty$.

It is well known that in a self-financed manner, the wealth process $x(.)$, under an admissible control $\pi(.)$, satisfies

(2) $$dx(t) = [rx(t) + \sum_{i=1}^{n}(b_i - r)\pi_i(t)]dt + \sum_{j=1}^{m}\sum_{i=1}^{n}\sigma_{ij}\pi_i(t)dW_j(t), \quad x(0) = x_0 .$$

In the control terminology $x(.)$ is the state process under the control $\pi(.)$. Note that $\pi_0(t) = x(t) - \sum_{i=1}^{n}\pi_i(t)$ is the amount invested in the bond, which is uniquely determined by $\pi(.)$ via the above equation. We define

$$b = (b_1 - r, ..., b_n - r)^T, \qquad \sigma = (\sigma_{ij})_{mxm}, \qquad \Gamma = \sigma\sigma^T .$$

Moreover, let σ_n denote the n x m matrix which is identical to the matrix consisting of the first n rows of σ, and let $\Gamma_n = \sigma_n \sigma_n^T$.

The dynamics in (2) can be rewritten as follows:

$$dx(t) = [rx(t) + b^T\pi(t)]dt + \pi^T\sigma_n dW(t), \qquad x(0) = x_0 .$$

Our objective is

$$\min \quad E\int_{t_0}^{\infty} e^{-2\rho t}[x(t) - x_0 e^{\mu t}]^2 dt ,$$

where $2\rho > 0$ is a discount factor. At this point we simply remark that ρ is introduced to guarantee the stabilizability of the control system, its actual value will have minimal impact on the result.

Applying a transformation of variables

$$y(t) = e^{-\rho t}[x(t) - x_0 e^{\mu t}], \qquad \tilde{\pi}(t) = e^{-\rho t}\pi(t)$$

to turn the above control problem into the following equivalent form:

$$\min \quad E\int_{t_0}^{\infty} | y(t) |^2 dt$$

subject to:

$$dy(t) = [(r - \rho)y(t) + b^T\tilde{\pi}(t) + (r - \mu)x_0 e^{(\mu - \rho)t}]dt + \tilde{\pi}(t)^T\sigma_n dW(t)$$

$$y(0) = 0 .$$

The above is a control problem to minimize a quadratic cost functional, with the system dynamics being linear with a nonhomogeneous term with respect to the state and control variables. Moreover, the system dynamics are stochastic. Hence, this is a SLQ problem. In order to relate the above control problem in (1) we can follow Yao et al. (Yao, Zhang & Zhou, 2006). Yao et al. (Yao, Zhang & Zhou, 2001) have investigated the SLQ model (1) in case k=1. Further on, they have extended (Yao, Zhang & Zhou, 2006) such type models and they have proposed a new approach to tracking either a given fixed growth rate or a stochastic market index. Both problems have been formulated as SLQ models.

Consider the introduced canonical formulation (1) of the above indefinite SLQ problem. To solve the SLQ problem (1) it is necessary to solve the following Riccati equation (Yao, Zhang & Zhou, 2006):

$$\mathrm{R(X)} := A^T X + XA + Q + \sum_{j=1}^{n} C_j^T X C_j$$

(3) $$- (XB + \sum_{j=1}^{n} C_j^T X D_j)[R + \sum_{j=1}^{n} D_j^T X D_j]^{-1}(XB + \sum_{j=1}^{n} C_j^T X D_j)^T = 0 .$$

with the additional condition $R+\sum_{j=1}^{n} D_j^T XD_j \succ 0$ (positive definite) for the unknown matrix X. The new equation has the inverse matrix depending on the unknown X and the additional strictly positive definiteness condition for the inverse one. If $\tilde{X}$ is the maximal positive definite solution of the above equation with $R+\sum_{j=1}^{n} D_j^T \tilde{X} D_j \succ 0$, then

$$\tilde{u}(t) = -(R+\sum_{j=1}^{n} D_j^T \tilde{X} D_j)^{-1}(\tilde{X}B+\sum_{j=1}^{n} C_j^T \tilde{X} D_j)^T \tilde{x}(t)$$

is an optimal state feedback control for (1). The optimal control $\tilde{u}(t)$ depending on the matrix $\tilde{X}$ which is the maximal solution to (3). There are few iterative algorithms for solving a generalized Riccati equation (3) under the assumption that R is a positive definite matrix. Very interesting the case where R is an indefinite symmetric matrix. We adapt the Newton-type algorithm for solving (3) and an algorithm that is called the Lyapunov iteration for (3) can be considered. Numerical simulations are used to demonstrate the performance of considered solvers.

Thus, following the classical linear quadratic theory we know that the following optimization problem is associated with the equation R(X) =0 , for example see Yao et al. (Rami & Zhou, 2000; Yao, Zhang & Zhou, 2006):

$$
\begin{aligned}
&\max < I_p, X > \\
(4)\quad &\begin{pmatrix} R+\sum_{j=1}^{n} D_j^T XD_j & (XB+\sum_{j=1}^{n} C_j^T XD_j)^T \\ XB+\sum_{j=1}^{n} C_j^T XD_j & Q+A^T X + XA+\sum_{j=1}^{n} C_j^T XC_j \end{pmatrix} \succ= 0 \\
&R+\sum_{j=1}^{n} D_j^T XD_j \succ 0 \\
&X \succ 0,
\end{aligned}
$$

where <X, Y> denotes the matrix inner-product. The above convex optimization problem is called a semidefinite programming problem. We use the existing MATLAB functions for solving the semidefinite programming problem. The solvability of R(X) = 0 and the corresponding semidefinite programming problem and connections between the maximal positive definite solution to R(X) = 0 and the positive definite solution to (4) are fully investigated in (Rami, Zhou & Moore, 2000; Rami & Zhou, 2000). The obtained results are related to R(X)=0 where n=1. In this special case the equation R(X) = 0 is solvable if and only if the LMI (4) (n=1) with $X \succ 0$ are feasible. We cite the following theorem (Theorem 10, Rami & Zhou, 2000) where it is claimed that if equation (3) (n=1) has a maximal positive definite solution then it is the unique optimal solution to the related semidefinite programming problem. We can extend this conclusion to our consideration. The above conclusion stay valid in more general case, i.e. if rational matrix equation (3) with n>1 has a maximal positive definite solution then it is the unique optimal solution to the related semidefinite programming problem (4). In practical, it is interesting to find the solvability margin r^* of (3). The solvability margin is defined as the largest the nonnegative scalar $r \geq 0$ such that (3) has a solution for any symmetric matrix R with $R > -r^* I$. It is easy to extend Theorem 11 derived from Rami & Zhou (Rami & Zhou, 2000) for the equation (3) in general case (n>1).

Theorem 1. The solvability margin r^* can be obtained by solving the following semidefinite programming problem:

$$
\begin{aligned}
&\min(-r) \\
&\begin{pmatrix} A^T X + XA+\sum_{j=1}^{n} C_j^T XC_j + Q & XB+\sum_{j=1}^{n} C_j^T XD_j \\ (XB+\sum_{j=1}^{n} C_j^T XD_j)^T & \sum_{j=1}^{n} D_j^T XD_j - rI \end{pmatrix} \succ= 0 \\
&\sum_{j=1}^{n} D_j^T XD_j - rI \succ 0 \\
&r > 0.
\end{aligned}
$$

The margin r^* (Rami & Zhou, 2000) has the properties:

If the smallest eigenvalue of $R(\lambda_{\min}(R))$ is such that $\lambda_{\min}(R) > -r^*$, then (3) has a solution.

If the largest eigenvalue of $R(\lambda_{\max}(R))$ is such that $\lambda_{\max}(R) \leq -r^*$, then (3) has no solution.

We have seen that the feasibility of LMIs is necessary and sufficient for the solvability of (3).

Numerical Solution of the Generalized Riccati Equation

Yao et al. (Yao, Zhang & Zhou, 2006) have considered the application the LMI techniques for solving the SLQ model (1). This techniques is presented via LMI problem (4). Here we propose two recursive equations for solving equation (3). These iterations can be considered as an effective alternative to (4). First, the Newton method for solving equation (3) is considered. The Newton method to the rational matrix equation R(X)=0 can be applied under the conditions that R(X) is stabilizable and that the inequality R(X)$\geq$0 is solvable in $domR = \left\{ X = X^T, R + \sum_{j=1}^{n} D_j^T X D_j > 0 \right\}$. Under these conditions, Damm and Hinrichsen (Damm & Hinrichsen, 2001) have proved the convergence of Newton's method if the method starts at any stabilizing initial point X_0. The standard Newton-iteration for equation R(X) = 0 has the following form

$$X_{i+1} = X_i - (R'_{X_i})^{-1}(R(X_i)), \quad i=0,1,2, \ldots$$

where R'_{X_i} is known as the Frechet derivative of R(X) at X_i. The Newton algorithm becomes

(5)
$$(A + F_{X_i})^T X_{i+1} + X_{i+1}(A + F_{X_i}) + Q + F_{X_i}^T R F_{X_i} + \sum_{j=1}^{n} (C_j + D_j F_{X_i})^T X_{i+1}(C_j + D_j F_{X_i}) = 0,$$

where $F_X = -(R + \sum_{j=1}^{n} D_j^T X D_j)^{-1}(XB + \sum_{j=1}^{n} C_j^T X D_j)^T$. The following theorem is derived by Damm & Hinrichsen:

Theorem 2 (Theorem 6.1**,** Damm & Hinrichsen, 2001). Assume that there exist a solution $\tilde{X} \in domR$ to R(X) $\geq$ 0 and a stabilizing matrix X_0 (i.e. R'_{X_0} has eigenvalues in the open left plane). Then the iteration scheme (5) defines a sequence $\{X_i\}$ in $domR$ with the following properties:

(i) for i = 1, 2, . . . : $X_i \geq X_{i+1} \geq \tilde{X}$ and R(X_i)$\leq$0;

(ii) for I = 0, 1, 2, . . . : R'_{X_i} is stable;

(iii) $\{X_i\}$ converges to a limit matrix $X_\infty \in domR$ that satisfies R(X_∞)=0;

(iv) X_∞ is the greatest solution of R(X)$\geq$0 and all eigenvalues of R'_{X_∞} lie in the closed left plane.

In the last equation we replace X_{i+1} with X_i in the expression $\sum_{j=1}^{n} (C_j + D_j F_{X_i})^T X_{i+1}(C_j + D_j F_{X_i})$ and we derive the new formula for the Lyapunov iteration to solve R(X)=0, which is

(6)
$$(A + F_{X_i})^T X_{i+1} + X_{i+1}(A + F_{X_i}) + Q + F_{X_i}^T R F_{X_i} + \sum_{j=1}^{n} (C_j + D_j F_{X_i})^T X_i (C_j + D_j F_{X_i}) = 0.$$

The convergence properties of iteration (6) in case n=1 is derived in the following theorem:

Theorem 3 (Theorem 2.10, Ivanov, 2007). Assume there exist Hermitian matrices $\hat{X}$ and X_0 such that R($\hat{X}$)$\geq$0 and $X_0 > \hat{X}$, R(X_0)<0 and $A + F_{X_0}$ is stable. Then for the matrix sequence $\{X_i\}$ defined by (6) are satisfied:

(i) $X_i \geq X_{i+1}$, $X_i \geq \hat{X}$, R(X_i) < 0, i = 0, 1, 2, . . .;

(ii) $A + F_{X_i}$ is stable for i = 0, 1, 2, . . .;

(iii) $\lim_{i\to\infty} X_i = \tilde{X}$ is a solution of R(X) **=** 0 with $\tilde{X} > \hat{X}$. Moreover, if $X_0 > X$ for all solutions X of R(X) **=** 0, then $\tilde{X}$ is the maximal solution;

(iv) the eigenvalues of $A + F_{\tilde{X}}$ lie in the closed left half plane. In addition, if R($\hat{X}$) > 0, then all eigenvalues of $A + F_{\tilde{X}}$ lie in the open left half plane.

In our model of portfolio optimization the matrix *R* can be a negative definite or even a zero matrix. In such cases the expression $R + \sum_{j=1}^{n} D_j^T X D_j$ depends on the unknown matrix *X* and can be singular, so $(R + \sum_{j=1}^{n} D_j^T X D_j)^{-1}$ is not defined and therefore the Newton iterations (5) and Lyapunov iterations (6) are not applicable. The only working method in such cases is the optimization problem (4), but reaching of the optimal solution is not guaranteed when *R* is negative definite (Rami & Zhou, 2000). Moreover in our previous works (Ivanov & Lomev, 2009), (Ivanov, Lomev & Netov, 2010) we demonstrated that methods (5) and (6) are faster when R is a positive definite matrix. Therefore if we can find a transformation of $R(X)$ where instead of R we have new symmetric matrix $\tilde{R}$ that is a positive definite matrix, then we might expect improvement of the numerical properties of the solution. There are many examples of such transform, for instance proposed in (Lin, Bao & Wei 1994). Let's introduce new variable $X=Z+Y$ in (3):

$$R+\sum_{j=1}^{n} D_j^T X D_j = R+\sum_{j=1}^{n} D_j^T (Z+Y) D_j = R+\sum_{j=1}^{n} D_j^T Z D_j + \sum_{j=1}^{n} D_j^T Y D_j = \tilde{R}+\sum_{j=1}^{n} D_j^T Y D_j$$

The Z matrix can be selected in a way to assure that $\tilde{R}$ is a positive definite matrix. After transformation we obtain new form of (4), where the unknown variable is $(Y=Y^T)$:

$$\max < I_p, Y >$$

$$(7) \qquad \begin{pmatrix} \tilde{R}+\sum_{j=1}^{n} D_j^T Y D_j & (L^T+YB+\sum_{j=1}^{n} C_j^T Y D_j)^T \\ L^T+YB+\sum_{j=1}^{n} C_j^T Y D_j & \tilde{Q}+A^T Y+YA+\sum_{j=1}^{n} C_j^T Y C_j \end{pmatrix} >= 0$$

$$\tilde{R}+\sum_{j=1}^{n} D_j^T Y D_j > 0$$

where $\tilde{Q} = A^T Z + ZA + Q + \sum_{j=1}^{n} C_j^T Z C_j$ and $L = B^T Z + \sum_{j=1}^{n} D_j^T Z C_j$.

Numerical Simulations

Our experiments are executed in MATLAB on 2,16GHz PENTIUM(R) Dual CPU computer. We denote tol - a small positive real number denoting the accuracy of computation, $E_s = \| R(X_s) \|_2$, It - number of iterations for which the inequality $E_{It} \le tol$. The last inequality is used as a practical stopping criterion.

The coefficients of (3) will be generated as a pseudo-random numbers. All experiments will be carried out with negative definite matrix R and for different values of the dimension parameter p we shall generate series of 100 simulations. For each of the series the maximum number of iterations (mIt) and the average number of iterations (avIt) for finding of the solution are calculated. The details of the test simulation are :

R= diag [-0.001, -0.5], q=2; A=randn(p,p)/100-0.5 I_p, B=2randn(p,2), C_1=randn(p,p)/10; D_1=2 randn(p,2);

The selected transformation is: Z= 0.4 I_2. The results are presented in the following table:

Table 1: Maximum and average number of iterations for simulated 100 cases

	LMI(4):		**NI (5):**		**LI (6):**		**LMI(7):**	
p	**m It**	**av It**	**m It**	**av It**	**m It**	**Av It**	**m It**	**av It**
10	55	39.3	6	4.1	20	14.6	31	26.7
12	54	41.7	6	4.2	21	14.6	31	25.7
15	68	57.3	6	4.2	22	15.8	27	27.2
The total time for solving of 100 cases (in seconds)								
15	111.42		4.93		1.79		40.46	
20	320.46		118.06		20.4		118.46	

Conclusion

The obtained results confirm that the introduction of new variable leads to substantial improvement of LMI method. Again the Lyapunov approach (6) is the fastest method.

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