An optimal control problem by controlling heat source of the surface of tissue

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Abstract: A distributed optimal control problem for a system described by bio-heat equation for a homogeneous plane tissue is analytically investigated such that a desired temperature of the tissue at a particular point of location of tumour in hyperthermia can be attained at the end of a total time of operation of the process due to induced microwave on the surface of the tissue which is taken as control. Here the temperature of the tissue along the length of the tissue at different times of operation of the process are numerically calculated which display the rise of the desired temperature of the tumour.

Keywords: Bio-heat equation, hyperthermia, optimization, microwave, tumour, control.

Introduction

Computer simulation plays a vital role in treating rise of temperature of tumour to it's therapeutic value by means of optimal distributions of the applied heat source and surface cooling temperature. In this respect, the consideration of physiological responses of the patient at the time of hyperthermia treatment, the region of the tissue affected by tumour, the anatomial feature of the treated patient and blood flow rates of the tissue should be taken into account with much importance for achieving the temperature of the tumour to it's therapeutic value avoiding the damage of healthy tissue due to overheating.

Deng and Liu (2002) investigated analytical solutions described by bio-heat transfer equation due to transient heating on the skin surface with the aid of Green's function . Dhar and Sinha (1989) carried out analytically a distributed optimal control problem in a multilayered tissue to attain desired rise of temperature of the tumour by controlling surface cooling temperature . Wagter (1986) made an important contribution on optimization in plane tissue by multiple electro – magnetic applicaitors. Butkovasky (1969) had studied the fundamentals of optimal control problems in distributed parameter system. Dhar and Sinha (1988) developed an optimal control problem analytically to attain desired temperature of the tumour by induced heat source at least possible time.

An analytical investigation was developed on computations for optimization problems in hyperthermia by finite difference method (Das et. al., 1999). Kowalski and Jin (2003) carried out analytically on optimization in hyperthermia by electro – magnetic annular phased arryays. In Loulou and Scott (2002) investigated a study on thermal dose optimization in hyperthermia using conjugate gradient method. Bagaria and Johnson (2005) studied analytically optimal control problem in bio-heat equation to achieve ideal hyperthermia condition using expliicit finite difference method. An analytical investigation was performed on optimization of radio – immunotherapy interations with hyperthermia in Kinuya et. al. (2004). In course of investigation on empirical dose construciton for oncological hyperthermia Szasz and Vincze (2006) developed Pennes equation by inducing the entire energy balance. Rapoport et. al. (2009) studied on chemotherapeutic intervention on tumours by ultrasound. Liu and Chen (2009) studied analytically the prediction of temperature in tissues described by bio-heat transfer problem in a bi-layered spherical tissue by considering blood perfusion and metabolism. Shih et. al. (2008) investigated the feasibility of heating on tumour by high intensity focussed ultrasound in thermal surgery.

Kuznetsov (2006) investigated optimal control problem to maximize temperature in the tumour at the end of time of the process due to spatial volumetric heat generation by assuming fixed total volumetric heat generation over the duration of the process. With the aid of conjugate gradient method, a distributed optimal control problem for a system described by bioheat equation in a homogeneous plane tissue due to induced microwave was investigated by Dhar and Dhar (2010) and Dhar et. al. (2012).

In this paper, a distributed optimal control problem described by bio-heat equation for a homogeneous tissue is analytically investigated such that a desired temperature of the tissue at a particular point of location of tumour can be attained at the end of total time of operation of the process by means of controlling induced microwave on the surface of the tissue when the surface cooling temperature is constant. Here the switching time during which the microwave power is operative has been obtained by using conjugate gradient method under calculus of variation.

A numerical temperature distributions of the tissue at different times on various values of total time of operation have been obtained which displays the rise of desired temperature of the tumour.

Mathematical Analysis

The one dimensional bio-heat equation (Deng 2002, Dhar 1989) can be written as,

$$\rho c \frac{\partial \chi}{\partial t} = k \frac{\partial^2 \chi}{\partial x^2} + \omega(\chi_a - \chi) + Q(t) + Q_m \tag{1}$$

Boundary condition :

$$k \frac{\partial \chi}{\partial x} = h\{ \chi - u(t) \} \quad on \quad x = 0$$
$$\chi = \chi_a \quad \text{on} \quad x = L \tag{2}$$

Initial Condition:

$$\chi(x, o) = \chi_0 \tag{3}$$

We would like to attain the desired temperature χ^* at the point $x = x_1$, where the tumour is located at the end of the total time T of the process by controlling optimally Q(t).

Thus the functional (Butkovasky 1969, Loulou 2002)

$$\frac{1}{2}\int_0^L \left\{\chi^* - \chi(x,T)\right\}^2 \delta(x-x_1)dx \tag{4}$$

is to be minimized.

The first term designates the square deviation of the temperature χ^* from $\chi(x,t)$ at $x = x_1$.

Let us write a functional J, given by (Butkovasky 1969, Loulou 2002)

$$\begin{aligned} J = &-\frac{1}{2}\int_0^L \left\{\chi^* - \chi(x,T)\right\}^2 \delta(x-x_1)\,dx \\ &+\int_0^L\int_o^T \psi(x,t)\left\{\frac{k}{\rho c}\frac{\partial^2\chi}{\partial x^2}+\frac{\omega}{\rho c}(\chi_a-\chi)+\frac{1}{\rho c}Q(t)+\frac{Q_m}{\rho c}-\frac{\partial}{\partial t}\chi\right\}dxdt \end{aligned} \tag{5}$$

where $\psi(x,t)$ is the auxiliary function.

By considering Q_m as constant, the first variation of the function J can be written as,

$$\begin{aligned} \delta J = &\int_0^L \left\{\chi^* - \chi(x,T)\right\}\delta(x-x_1)\,\delta\chi(x,T)\,dx \\ &+\frac{k}{\rho c}\int_o^T \psi(L,t)\,\delta\chi_x(L,t)\,dt + \frac{1}{\rho c}\int_o^T\left\{k\frac{\partial\psi}{\partial x}(o,t)-h\psi(o,t)\right\}\delta\chi(o,t)\,dt \\ &+\frac{h}{\rho c}\int_o^T\psi(o,t)\,\delta u(t)dt - \frac{k}{\rho c}\int_o^T\frac{\partial}{\partial x}\psi(L,t)\,\delta\chi(L,t)dt + \frac{k}{\rho c}\int_o^T\int_o^L\frac{\partial^2}{\partial x^2}\psi(x,t)\,\delta\chi(x,t)\,dxdt \\ &-\frac{\omega}{\rho c}\int_o^L\int_o^T\psi(x,t)\,\delta\chi(x,t)\,dxdt + \frac{1}{\rho c}\int_o^T\int_o^L\psi(x,t)\,\delta Q(t)\,dxdt \\ &+\int_0^L\int_o^T\frac{\partial\psi(x,t)}{\partial t}\,\delta\chi(x,t)\,dxdt - \int_o^L\psi(x,T)\,\delta\chi(x,T)\,dx \\ &+\int_0^L\psi(x,o)\,\delta\chi(x,o)\,dx \end{aligned} \tag{6}$$

with the help of equations (2) and (3). By assuming δJ to vanish for any $\delta\chi_x(L,t)$, $\delta\chi(x,t)$, $\delta\chi(o,t)$, $\delta\chi(x,T)$, $\delta Q(t)$, $\delta u(t)$ and taking $\delta\chi(x,o)$, $\delta\chi(L,t)$ both equal to zero, a system of auxiliary function $\psi(x,t)$ is obtained as,

$$\frac{\partial\psi}{\partial t} + \frac{k}{\rho c}\frac{\partial^2\psi}{\partial x^2} = \frac{\omega}{\rho c}\psi. \tag{7}$$

$$k\frac{\partial\psi}{\partial x} = h\psi \quad \text{on} \quad x = 0 \tag{8}$$

$$\psi(x,t) = 0 \quad on \quad x = L$$

$$\psi(x,T) = \{\chi^* - \chi(x,T)\}\,\delta(x - x_1) \tag{9}$$

and the optimal values of the controls Q(t) and u(t) stand,

$$\begin{aligned} Q(t) &= Sign\,\frac{1}{\rho c}\int_0^L \psi(x,t)\,dx \\ u(t) &= Sign\;\psi(0,t), \end{aligned} \tag{10}$$

Here the conjugate gradient method with the aid of calculus of variation has been used (Butkovasky 1969, Loulou 2002). Considering $\chi_1(x,t) = \chi\ (x,t) - \chi_a$ and expressing $\chi_1(x,t)$ in Finite Sine Transform, given by,

$$\overline{\chi}_{1n}(t) = \int_0^L \chi_1(x,t)\sin p_n(L-x)\,dx \tag{11}$$

and

$$\chi_1(x,t) = \sum_{n=1}^{\infty} \overline{\chi}_{1n}(t) \times \frac{2\sin p_n(L-x)}{L - \dfrac{\sin 2p_n L}{2p_n}} \tag{12}$$

where p_n are positive, real roots of the equation,

$$p\,cot(pL) = \frac{-h}{k} \tag{13}$$

the equation (1) with the help of equations (2), (3) and (13) stands,

$$\frac{d}{dt}\overline{\chi}_{1n}(t) + \alpha_{1n}\overline{\chi}_{1n}(t) = \alpha_{3n}Q(t) + \alpha_{4n} + \alpha_{5n}; \quad n = 1,2,3,\cdots \tag{14}$$

where,

$$\begin{aligned} \alpha_{1n} &= \frac{1}{\rho c}\left\{k\,p_n^2 + \omega\right\}, \\ \alpha_{4n} &= \frac{h}{\rho c}\left\{u(t) - \chi_a\right\}\sin p_n L, \\ \alpha_{3n} &= \frac{1}{\rho c}\left(\frac{1-\cos p_n L}{p_n}\right) \\ \alpha_{5n} &= \frac{1}{\rho c}\left(\frac{1-\cos p_n L}{p_n}\right)Q_m \end{aligned} \tag{15}$$

Finally we get,

$$\chi(x,t) = \chi_a + \sum_{n=1}^{\infty}\overline{\chi}_{1n}(t) \times R_n(x) \tag{16}$$

The solution of equation (14) with the help of equation (15) stands,

$$\overline{\chi}_{1n}(t) = [(\chi_o - \chi_a)\left(\frac{1-\cos p_n L}{p_n}\right) + \frac{h}{\rho\, c}\sin p_n L \int_0^t \{u(\xi) - \chi_a\} e^{\alpha_{1n}\xi} d\xi$$

$$+ \left(\frac{1-Cos p_n L}{p_n}\right)\frac{1}{\rho\, c} Q_m \int_0^t e^{\alpha_{1n}\xi} d\xi \tag{17}$$

$$+ \frac{(1-\cos p_n L)}{p_n}\frac{1}{\rho c}\int_0^t Q(\xi) e^{\alpha_{1n}\xi} d\xi] \times e^{-\alpha_{1n}t}; \quad n = 1,2,3,\cdots$$

where

$$R_n(x) = \frac{2\sin p_n(L-x)}{L - \dfrac{\sin 2p_n L}{2p_n}} \tag{18}$$

The corresponding solution of equation (7) with the help of equations (8) and (9) can be written as, with the help of earlier Finite Transform,

$$\psi(x,t) = \sum_{m=1}^{\infty} \overline{\psi}_m(t)\, R_m(x) \tag{19}$$

where

$$\overline{\psi}_m(t) = \{(\chi^* - \chi_a) - \sum_{n=1}^{\infty} \overline{\chi}_{1n}(T) \times R_n(x_1)\}\, Sin\, p_m(L - x_1) \times e^{-\alpha_{1m}(T-t)} \tag{20}$$

for p_m are roots of the equation (13).

Considering u(t) as constant, the value of optimal control Q(t) can be obtained from equation (10) with the help of equations (17) , (18) , (19) and (20).

Here we have assumed that the time dependent Q(t) (Wm^{-3}) is only controllable input variable which is piecewise constant function of time that changes value at certain specified discrete instants considered as switching times (Wagter, 1986).

For the sake of simplicity we consider only one specified switching time $t = t_1$. Thus, according to equation (10) one can write

$$Q(t_1) = \int_0^L \psi(x,t_1)dx = 0 \tag{21}$$

where Q(t) assumes two extreme values in $(0,t_1)$ and (T,t_1), as one considers Q(t) a singular control, which can be obtained with the help of equations (16) - (21) by means of simulation.

Results and Discussions

Data used in computation are given as follows :

c	=	3770 J $kg^{-1}\,{}^{0}C^{-1}$
ρ	=	998 kgm^{-3}
k	=	.5 $Wm^{-1}\,{}^{0}C^{-1}$
h	=	6 $Wm^{-2}\,{}^{0}C^{-1}$
χ_a	=	$37^{0}C$
χ^{*}	=	$43^{0}C$
L	=	.01 m,
x_1	=	.006m
ω	=	3000 $Wm^{-3}\,{}^{0}C^{-1}$
Q_m	=	33800 Wm^{-3}
χ_0	=	$25^{0}C$
T	=	600s, 800s, 1000s
u(t)	=	$20^{o}C$

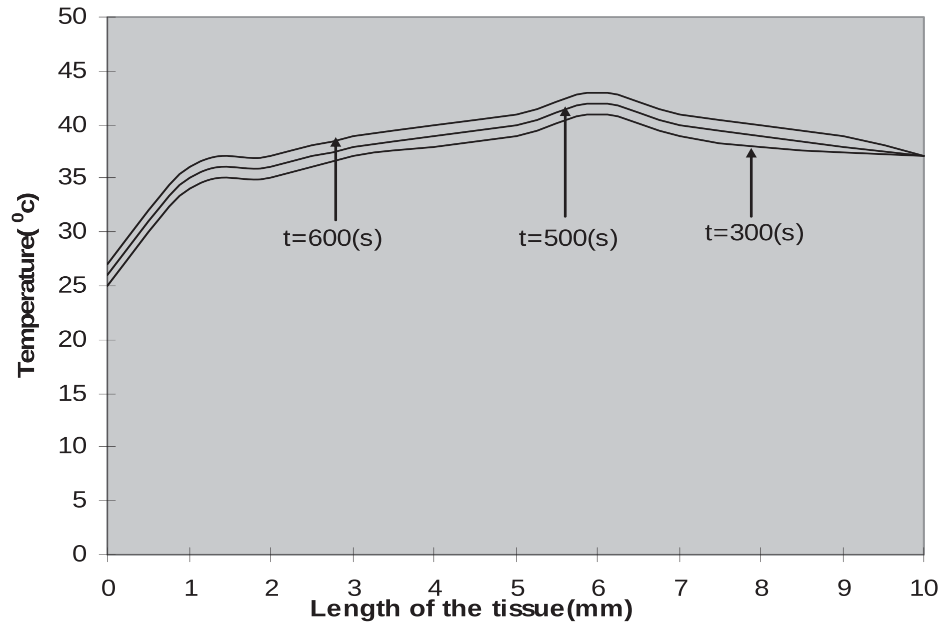

Fig 1 : Temperature(^{0}c) distribution of the tissue along the length of the tissue(mm) for Q(t)=338083 wm^{-3}, 0<=t<=500; Q(t)=9857 wm^{-3} , 500< =t<=600;

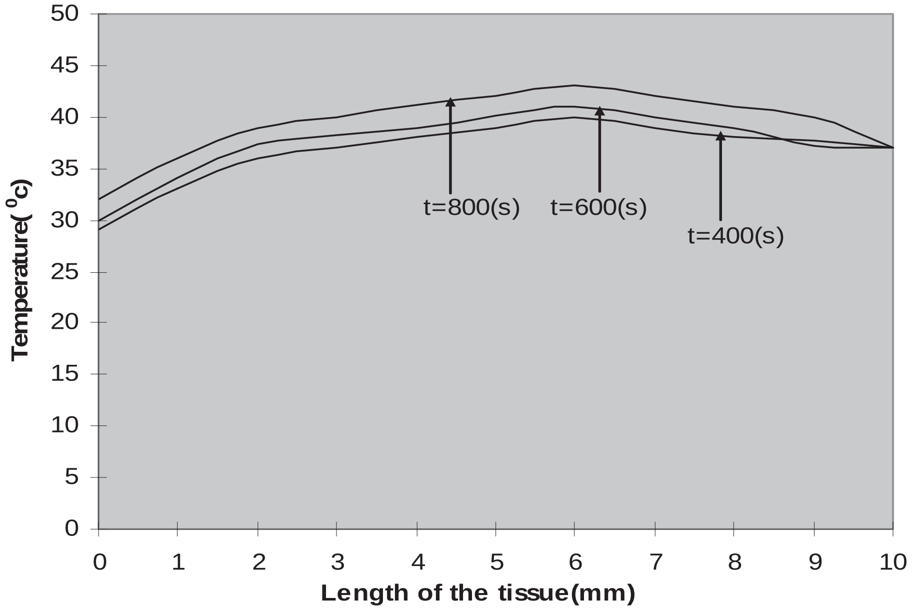

Fig 2 : Temperature(0c) distribution of the tissue along the length of the tissue(mm) for Q(t)=316053 wm^{-3}, 0<=t<=700; Q(t)=10512 wm-3 , 700< =t<=800;

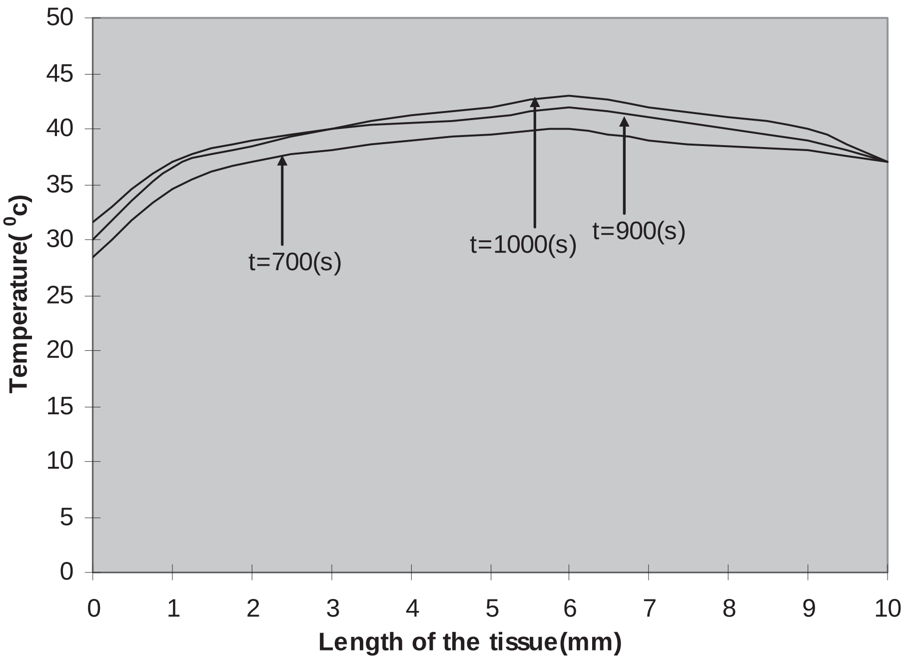

Fig 3 : Temperature(^{0}c) distribution of the tissue along the length of the tissue(mm) for Q(t)=305144 wm^{-3}, 0<=t<=900; Q(t)=10823 wm^{-3} , 900< =t<=1000;

Fig 1 displays the temperature of the tissue along the length of the tissue for $Q(t) = 338083$ Wm^{-3} , $0 \le t \le 500$; $Q(t) = 9857$ Wm^{-3} , $500 \le t \le 600$. Fig 2 depicts the temperature of the tissue along the length of the tissue subject to $Q(t) = 316053$ Wm^{-3} , $0 \le t \le 700$; $Q(t) = 10512$ Wm^{-3} , $700 \le t \le 800$. In Fig 3 the temperature of the tissue along it's length due to the application of optimal volumetric heat generation rate $Q(t) = 305144$ Wm^{-3} , $0 \le t \le 900$; $Q(t) = 10823$, $900 \le t \le 1000$. It is observed that desired temperature 43^0C at the particular tumour point $x_1 = .006$m is attained at the end of operation of the process T = 600s , 800s and 1000s in Fig 1 , Fig 2 and Fig 3 respectively.

Further it requires mentioning that as the total time of operation of the process increases from T =600s to 1000s, the switching time increases with the decrease of Q(t) in the first time segment of operation and corresponding increase of Q(t) in the second time segment of operation. Again the temperature of the tissue on left side of the tumour steadily increases and attains the desired temperature 43^oC on the point of tumour at the end of the process. On the right side of the tumour, the temperature rapidly decreases to arterial temperature 37^0C till the end of the process as we consider the cases at T = 600s , T = 800s and T = 1000s displayed in Fig 1, Fig 2 and Fig 3 respectively. Thus the temperature of the healthy tissue is not been overheated above 43^oC.

Conclusion

This analytical study may be extended for further developments considering different times of operation and also different locations of the tumour having various lengths of the tissue.

It is to note that in the paper [Dhar and Dhar, 2010] the desired tumour temperature is attained within the total time of the operation of the process (switching time t_1 (say)). Here, the microwave is switched off during the second time segment (t_1, T) . But, in this paper the desired temperature of the tumour is attained at the end of operation of the process at time T. In this case the microwave is not switched off but its intensity is substantially reduced in the second time segment (t_1, T).

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Nomenclature:

c	=	specific heat of tissue, J/kg ^oC
h	=	heat transfer coefficient between the skin and the ambient air, $Wm^{-2}/^0C$
k	=	thermal conductivity of tissue, $W\,m^{-1}/^0C$
L	=	length of the tissue, m
x_1	=	location of the tumour, m
χ	=	temperature, ^0C
χ_a	=	arterial temperature, ^0C
χ_0	=	initial temperature, ^0C
u(t)	=	temperature of the surrounding medium, ^0C
χ^*	=	desired temperature to be attained, ^0C
T	=	Total time of the process, s
t_1	=	switching time, s
Q(t)	=	optimal heat generation rate due to volumetric heating, Wm^{-3}
ρ	=	density of tissue, kg m^{-3}
δ	=	dirac – delta function.
ω	=	product of flow and heat capacity of blood, W m^{-3} / ^0C
Q_m	=	rate of metabolic heat generation, Wm^{-3}

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